Divisors

Wednesday, November 13, 2024 9:11 AM

Today X stands for an integral (separated, irreducible) noetherian scheme.

Def: - The group of Weil divisors is the additive free abelian subgp spanned by the irreducible closed subsets of codim 1. It's denoted $\operatorname{Weil}(X)$.

. A member of the group of Weil divisors is called a Weil divisor. So a Weil divisor is a formal sum $\sum_{i=1}^{t} n_i Z_i$, where Z_i irr and $\operatorname{codim}_X(Z_i)=1$, $n_i \in \mathbb{Z}$

. $\sum_{i=1}^{t} n_i Z_i + \sum_{i=1}^{t} n_i' Z_i = \sum_{i=1}^{t}(n_i+n_i')Z_i$

Eg: In $\mathbb{P}^1_{\mathbb{C}}$, codim 1 pts are closed pts. So a Weil divisor looks like $\sum_i n_i([x_i:y_i])$, such as $[1:0]-[0:1]$.

Eg. In $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[x,y])$ $\quad V(x)-3V(y)$ is a Weil div.

Eg. Take the rational function $x/y^3 \in \mathbb{C}(x,y)$ on $\mathbb{A}^2_{\mathbb{C}}$. $V(x)-3V(y)$ reads the zeros and poles with order of this rational function.

For today unless otherwise mentioned we shall assume X is regular in codimension 1. This means for every $\xi \in X$ such that $\dim \mathcal{O}_{X,\xi}=1$, $\mathcal{O}_{X,\xi}$ is a regular ring.

If Z is irr of codim 1 and η_Z be the generic point of Z (i.e. $\overline{\{\eta_Z\}}=Z$), $\dim \mathcal{O}_{Z,\eta_Z}=1$.
Indeed, recall for $x \in X$, $\operatorname{codim}_X(\overline{\{x\}})=\dim \mathcal{O}_{X,x}$.
The codim of a point $x\in X$ is by definition $\dim \mathcal{O}_{X,x} = \operatorname{codim}_X(\overline{\{x\}})$.

§: Divisor of a rational function.

Recall: $K(X)$ = Field of rational function of X
$= \mathcal{O}_{X,\eta}$ where η is the generic pt of X.
$= \{ f \in \mathcal{O}_X(U) \mid U \underset{\text{open}}{\subseteq} X \} \big/ \; f\in\mathcal{O}_X(U) \sim \tilde{f} \in \mathcal{O}_X(\tilde{U})$ iff $\exists V \subseteq U\cap\tilde{U}$ s.t. $f|_V = \tilde{f}|_V$

Given an $Z \underset{\text{codim 1, irr closed}}{\subset} X$, since the local ring $\mathcal{O}_{X,\eta_Z}$ (η_Z gen pt of Z) is reg of dim 1, the maximal ideal is gen by 1 elt. Choose a gen f. For is reg of dim 1, The maximal ideal is gen by 1 elt. Choose a gen f. For $\alpha \in \mathcal{O}_{X,\eta}$, define

$$\boxed{\operatorname{ord}_Z(\alpha) = \max\{n \mid \alpha \in (f^n)\}}$$

note, $\operatorname{ord}_Z(\alpha)$ does not depend on the choice of the gen f, as any other gen is $f \cdot u$ where u is a unit.

Since $K(X) = \operatorname{Frac}(\mathcal{O}_{X,\eta_Z})$, extend ord_Z to $K(X) \longrightarrow \mathbb{Z}$ by setting $\operatorname{ord}_Z(\alpha/\beta) = \operatorname{ord}_Z(\alpha) - \operatorname{ord}_Z(\beta)$ for $\alpha, \beta \in \mathcal{O}_{X,\eta_Z}$. and $\operatorname{ord}_Z(0) = \infty$.

Prop / Def. For $f \in K(X)^*$,

$$\sum_{\substack{Z \text{ irr of} \\ \text{codim } 1}} \operatorname{ord}_Z(f) \cdot Z \quad \text{is an Weil divisor.}$$

This is called the divisor of f and denoted $\operatorname{div}(f)$.

Pf. Since X is smooth, choose a finite affine open cover $X = \bigcup_{i=1}^{r} U_i$.

For each i, write $f = f_i/g_i$. For an irr closed subset Z of codim 1 such that $Z \cap U_i \neq \phi$ or equivalently s.t the gen pt of Z, $\eta_Z \in U_i$,
$\operatorname{ord}_Z(f) \neq 0 \Rightarrow$ either $\operatorname{ord}_Z(f_i) \neq 0$ or $\operatorname{ord}_Z(g_i) \neq 0$.

Now for $h \in \mathcal{O}_X(U_i)$, and a codim 1 pt $p \in \operatorname{Spec}(\mathcal{O}_X(U_i)) = U_i$, $\operatorname{ord}_p(h) \neq 0$
$\Leftrightarrow h$ is not a unit in $\mathcal{O}_X(U_i)_p$
$\Leftrightarrow h \in p\,\mathcal{O}_X(U_i)_p$
$\Leftrightarrow h \in p$ in $\mathcal{O}_X(U_i)$

Now by Krull's Thm The set
$\{q \in \operatorname{Spec}(\mathcal{O}_X(U_i)) \mid h \in q\}$ is finite since each such prime is a min prime of $\mathcal{O}_X(U_i)/h$.

So # of codim 1 pts in $\mathcal{O}_X(U_i)$ s.t $\operatorname{ord}_{\overline{\{\xi\}}}(f_i) \neq 0$ or $\operatorname{ord}_{\overline{\{\xi\}}}(g_i) \neq 0$ is finite.

For an irr closed $Z \subseteq X$, $\operatorname{ord}_Z(f) \neq 0$
$\Rightarrow \exists\, i \quad \operatorname{ord}_{Z \cap U_i}(f_i/g_i) \neq 0$ for some i
$\Rightarrow \sum \operatorname{ord}_Z(f)\, Z$ is a finite sum

Rmk. For a codim 1 irr subset Z and $f \in K(X)$, $\operatorname{ord}_Z(f)$ measures The order of zeros and poles f along Z. Indeed since $K(X) = \operatorname{Frac}(\mathcal{O}_{X,\eta_Z})$ and η_Z is a principal ideal (t).
$t^{\operatorname{ord}_Z(f)} \cdot u$ where $u \in \mathcal{O}_{X,\eta_Z}^*$

t along Z.

and n_Z is a principal ideal (t).

any $f = t^{\mathrm{ord}_Z(f)} \cdot u$ where $u \in \mathcal{O}^*_{X,\eta_Z}$

Proof: • For any irr closed Z of codim 1,

$\mathrm{ord}_Z(fg) = \mathrm{ord}_Z(f) + \mathrm{ord}_Z(g)$.

• Thus $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$.

• $\mathrm{div}(1) = 0$ • $\mathrm{div}(1/f) = -\mathrm{div}(f)$

• Thus $\{\mathrm{div}(f) \mid f \in K(X)^*\} \subseteq \mathrm{Weil}(X)$

is a subgp.

Def: The quotient gp $\dfrac{\mathrm{Weil}(X)}{\{\mathrm{div}(f) \mid f \in K(X)^*\}}$ is

called the Weil divisor class gp or simply the divisor class gp and denoted $\mathrm{Cl}(X)$.

Eg. 1) $\mathrm{Cl}(\mathbb{A}^n_k) = \{0\}$ where k is a field.

$\mathbb{A}^n_k = \mathrm{Spec}(k[x_1, \dots, x_n])$. Z irr closed of codim 1 $\Rightarrow \exists$ an irr pol f s.t

$Z = V(f)$. Note $\mathrm{div}(f) = Z$.

So $K(\mathbb{A}^n_k) \xrightarrow{\mathrm{div}} \mathrm{Weil}(\mathbb{A}^n_k)$ is surj.

Eg 2: $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$. Given a codim 1 irr closed Z, $\exists$ a homo irr pol f, such that

$Z = V(f)$, define $\deg Z = \deg f$.

Have a ~~om~~ . $\deg : \mathrm{Weil}(\mathbb{P}^n_k) \to \mathbb{Z}$. $(\deg(V(x_0)) = 1)$

Given $D \in \mathrm{Weil}(\mathbb{P}^n_k)$, write

$$D = \sum_{i=1}^{r} n_i V(f_i) - \sum_{j=1}^{m} m_j V(g_j) \text{ where}$$

$n_i, m_j > 0$.

$$\deg D = \sum_{i=1}^{r} n_i \deg(f_i) - \sum_{j=1}^{m} m_j \deg(g_j) = 0$$

$$\Leftrightarrow \deg\left(\prod_{i=1}^{r} f_i^{n_i}\right) = \deg\left(\prod_{j=1}^{m} g_j^{m_j}\right)$$

$$\Leftrightarrow h = \frac{\prod_{i=1}^{r} f_i^{n_i}}{\prod_{j=1}^{m} g_j^{m_j}} \in K(\mathbb{P}^n_k) \text{ and}$$

$$D = \mathrm{div}(h).$$

So deg induces an isom $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$.

$[V(x_0)]$ being a gen.

End of 15.11.24 lecture

Def. Y integral scheme. $\underline{K(Y)}$ is the constant sheaf $\underline{K(Y)}(U) = K(Y)$.

Def. Given $D \in \mathrm{Weil}(X)$, define the sheaf $\mathcal{O}_X(D) \subseteq \underline{K(X)}$

$\dots \geq 0\} \cup \{0\} \subseteq \underline{K(X)}(U) = K(X)$

sheaf $\underline{K(X)}$ ...

<u>Def.</u> Given $D \in \mathrm{Weil}(X)$, define the sheaf $\mathcal{O}_X(D) \subseteq \underline{K(X)}$ by $\mathcal{O}_X(D)(U) = \{f \in K(X)^* \mid (\mathrm{div}(f) + D)|_U \geq 0\} \cup \{0\} \subseteq K(X)(U) = K(X)$

where for $D' = \sum_i n_i Z_i \in \mathrm{Div}(X)$,

- $D'|_U = \sum_i n_i (Z_i \cap U) \in \mathrm{Div}(U)$; • $D' \geq 0 \overset{\text{def}}{\iff} n_i \geq 0 \ \forall i$.

<u>Prop.</u> $\mathcal{O}_X(D) \subseteq \underline{K(X)}$ is an additive subgp sheaf.

<u>Pf.</u> It is straightforward to check $\mathcal{O}_X(D)$ is a sheaf.

We verify $\mathcal{O}_X(D)(U) \subseteq K(X)$ is a subgp.

Let $f_1, f_2 \in \mathcal{O}_X(D)(U)$ and $Z \underset{\text{codim } 1}{\subseteq} X$ such that $Z \cap U \neq \emptyset$. Choose a generator of the maximal ideal $\mathcal{O}_{X,\eta_Z}$, denoted t.

$t^{\mathrm{ord}_Z(f_1)} \mid f_1$, $t^{\mathrm{ord}_Z(f_2)} \mid f_2$ in $\mathcal{O}_{X,\eta_Z}$.

Thus $t^{\min\{\mathrm{ord}_Z(f_1),\, \mathrm{ord}_Z(f_2)\}} \mid f_1 + f_2$

$\Rightarrow \mathrm{ord}_Z(f_1+f_2) \geq \min\{\mathrm{ord}_Z(f_1), \mathrm{ord}_Z(f_2)\}$.

$\Rightarrow$ If $(D + \mathrm{div} f_1)|_U \geq 0$, $(D + \mathrm{div}(f_2))|_U \geq 0$. Then $(D + \mathrm{div}(f_1+f_2))|_U \geq 0 \Rightarrow f_1 + f_2 \in \mathcal{O}_X(U)$

Since $\mathrm{ord}_Z(-f_1) = \mathrm{ord}_Z f_1$, $-f_1 \in \mathcal{O}_X(U)$.

Thus $(\mathcal{O}_X(D)(U), +)$ is a group.

<u>Rmk.</u> Say $D = \sum_{i=1} n_i Z_i$, $\mathrm{div} f = \sum_{i=1} m_i Z_i$, $m_i = \mathrm{ord}_{Z_i}(f)$.

$f \in \mathcal{O}_X(D)(U) \iff$ for each i s.t $Z_i \cap U \neq \emptyset$, $n_i + \mathrm{ord}_{Z_i}(f) \geq 0$

$\iff$ if $n_i \geq 0$, f can have a pole of order atmost n_i along Z_i, If $n_i \leq 0$, $\mathrm{ord}_{Z_i}(f) \geq -n_i$ i.e f must have a zero of ord at least n_i along Z_i

<u>Cartier divisors</u>: Y be an integral scheme

<u>Def.</u> • Y integral $\underline{K(Y)^*}$ is the constant sheaf of abelian gps $\underline{K(Y)^*}(U) = K(Y)^*$, under pdt.

- $\mathcal{O}_Y^*$ is the sub sheaf of abelian gps given by $\mathcal{O}_Y^*(U)$ = units of the ring $\mathcal{O}_Y(U)$.

- The group (abelian) of <u>Cartier divisors</u> is

$\mathcal{O}_Y(U) = \ldots$

Def. The group (abelian) of Cartier divisors is the group $\Gamma(Y, \underline{K(Y)^*}/\mathcal{O}_Y^*) = \mathrm{Cart}(Y)$

- An elt of $\Gamma(Y, \underline{K(Y)^*}/\mathcal{O}_Y^*)$ is called a Cartier divisor.

- There is a natural ~~map~~ gp homo.

$$K(Y)^* \longrightarrow \mathrm{Cart}(Y)$$

The quotient gp, denoted $\mathrm{CaCl}(Y)$, is called the Cartier divisor class group.

Rmk. Given $s \in \mathrm{Cart}(Y)$, $\exists$ an open covering $Y = \bigcup_{i \in I} U_i$ and $f_i \in K(Y)^*$ s.t $s|_{U}$ is represented by f_i. On $U_i \cap U_j$, $f_i = u_{ij} f_j$ ~~for~~ for some $u_{ij} \in \mathcal{O}_Y(U_i \cap U_j)^*$

$\boxed{\text{* } X \text{ noeth, integral, reg in codim 1.}}$

Def/Prop: Given Z irr closed of codim 1,

$$\left(\underline{K(X)^*}/\mathcal{O}_X^*\right)_{\eta_Z} \cong \frac{\underline{K(X)^*}_{\eta_Z}}{(\mathcal{O}_X^*)_{\eta_Z}} = \frac{K(X)^*}{\mathcal{O}^*_{X,\eta_Z}}$$

For $h \in \mathcal{O}^*_{X,\eta_Z}$, $\mathrm{ord}_Z(h) = 0$.

Thus $\mathrm{ord}_Z : K(X)^* \longrightarrow \mathbb{Z}$ factors through $K(X)^*/\mathcal{O}^*_{X,\eta_Z}$.

- Given $s \in \mathrm{Cart}(X)$, define $\boxed{\mathrm{ord}_Z(s) = \mathrm{ord}_Z(s_{\eta_Z} \in \frac{K(X)^*}{\mathcal{O}^*_{X,\eta_Z}})}$

- Given $s \in \mathrm{Cart}(X)$, define

$$\boxed{\mathrm{div}(s) = \sum \mathrm{ord}_Z(s) \cdot Z \in \mathrm{Weil}(X).}$$

- The following claim gives a way to 'think about' the sum in $\mathrm{div}(s)$ for $s \in \mathrm{Cart}(X)$ and justifies why the sum on the right hand is finite.

Claim:* Given $s \in \mathrm{Cart}(X)$, choose a finite open covering $X = \bigcup_{i=1}^{r} U_i$ such that for each i, $\exists f_i \in K(X)^*$ satisfying $s|_{U_i} = f_i$ in $\underline{K(X)^*}/\mathcal{O}_X^*(U_i)$

Then $\mathrm{div}(s)|_{U_i} = \mathrm{div}(f_i)|_{U_i}$

... such that $Z \cap U_i \neq \emptyset$.

Then $\text{div}(s)|_{U_i} = \text{div}(f_i)|_{U_i}$

Pf. For $Z \subseteq X$ irr closed codim 1 such that $Z \cap U_i \neq \emptyset$.

$\text{ord}_Z(s) = \text{ord}_Z(f_i)$. So the claim follows.

Prop. (i) $\text{div}: \text{Cart}(X) \to \text{Weil}(X)$ is a gp homo.

(ii) For $f \in K(X)^*$, $\text{div}(f \in \text{Cart}(X)) = \text{div}(f \in K(X)^*)$

So div induces a gp homo.

$\overline{\text{div}}: \text{CaCl}(X) := \dfrac{\text{Cart}(X)}{K(X)^*} \longrightarrow \text{Cl}(X)$

(iii) When X is normal, div and $\overline{\text{div}}$ are injective.

Def. • A Weil divisor D on X is called <u>locally principal</u> if $\exists$ a (finite) open covering $X = \cup U_i$ and $f_i \in K(X)^*$ such that $D|_{U_i} = \text{div} f_i|_{U_i}$ for each i.

• A Weil divisor D is called <u>effective</u> if $D = \sum_{i=1}^{r} n_i Z_i$ where $n_i \geq 0$

Prop 1) $\text{div}(\text{Cart}(X)) \subseteq$ locally principal Weil divisors.

2) When X is normal, the above containment is an equality.

Pf. 1) follows from claim $*$.

2) Given a locally principal Weil divisor D, choose a finite open covering $X = \bigcup_{i=1}^{r} U_i$ and $f_i \in K(X)^*$ such that $D|_{U_i} = \text{div} f_i|_{U_i}$. Since $\text{div} f_i|_{U_i \cap U_j} = \text{div}(f_j)|_{U_i \cap U_j}$
$= D|_{U_i \cap U_j}$

$\text{div}(f_i/f_j)|_{U_i \cap U_j} = 0$. Since X is <u>normal</u>

$f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^*$. So $\{f_i \in \underline{K(X)^*}/\mathcal{O}_X^*(U_i)\}_i$ glue to give a global section

$f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^*$ … $/\mathcal{O}_X$

glue to give a global section

$$s \in \Gamma(X, \underline{K(X)^*}/\mathcal{O}_X^*).$$

By claim $*$, $\operatorname{div}(s) = D$.

- We examine surjectivity of $\operatorname{Cart}(X) \xrightarrow{\operatorname{div}} \operatorname{Weil}(X)$.

The following Thm is crucial.

Thm. Let A be a noetherian ring.

A is a unique factorization domain

$\iff$ A is normal and $\operatorname{Cl}(\operatorname{Spec} A) = \{0\}$.

Pf. See Prop 6.2, Hart.

Thm. Let X be a normal, noetherian scheme

$\operatorname{Cart}(X) \xrightarrow{\operatorname{div}} \operatorname{Weil}(X)$ is surjective (eq. bijective)

$\iff$ X is locally factorial, i.e for every affine $U \underset{\text{open}}{\subseteq} X$

$\mathcal{O}_X(U)$ is a U.F.D

$\iff$ $\overline{\operatorname{div}} : \operatorname{CaCl}(X) \xrightarrow{\sim} \operatorname{Cl}(X)$.

Pf. We only verify $\Leftarrow$ of the top $\iff$.

We know $\operatorname{div}(\operatorname{Cart}(X)) =$ locally principal Weil divisors of X.

Given $D \in \operatorname{Weil}(X)$ and $U \underset{\text{affine}}{\subseteq} X$ open.

$D|_U \in \operatorname{Weil}(U)$. But the Thm above $\operatorname{Cl}(U) = \{0\}$,

So $\exists f \in K(U)^* = K(X)^*$ such that $D|_U = \operatorname{div} f|_U$.

So D is locally principal.

End of 20.11.24 Lecture

Start of 22.11.24 Lecture

Thm. When X is regular, $\operatorname{div}: \operatorname{Cart}(X) \xrightarrow{\sim} \operatorname{Weil}(X)$

and $\overline{\operatorname{div}} : \operatorname{CaCl}(X) \xrightarrow{\sim} \operatorname{Cl}(X)$.

§ Picard group and CaCl

The results of this section are true for any locally integral scheme, which are not necessarily

... integral scheme, which are not necessarily ... smooth.

Thm. Y be an integral scheme. Given a $\delta \in \mathrm{Cart}(Y)$, define
$$\mathcal{L}(\delta)(U) = \{ f \in K(Y)^* \mid f_x \delta_x \in \mathcal{O}_x(U) - \{0\} / \mathcal{O}^*_{Y,x} \} \cup \{0\}$$
$\forall x \in U$

$$\subseteq \underline{K(Y)}(U) = K(Y)$$

Then 1) $\mathcal{L}(\delta)$ is a sub $\mathcal{O}_Y$-mod sheaf of the sheaf of abelian grps $(\underline{K(Y)}, +)$.

2) $\mathcal{L}(\delta)$ is invertible

Pf 1) is straightforward.

2) Claim: Given $\delta \in \mathrm{Cart}(Y)$. Choose a covering $Y = \bigcup_{i \in I} U_i$ s.t. $\delta|_{U_i} = f_i$ in $\Gamma(U_i, K(Y)^*/\mathcal{O}_Y^*)$

Then $\mathcal{L}(\delta)|_{U_i} \xleftarrow{\sim} \mathcal{O}_{U_i}$.
$1/f_i \leftarrow 1$

In particular $\mathcal{L}(\delta)(U_i) = 1/f_i \, \mathcal{O}_Y(U_i)$ and
$\mathcal{L}(\delta)|_{U_i} \xrightarrow{\sim} 1/f_i \, \mathcal{O}_Y(U_i)$ when U_i is affine.

Pf- $1/f_i \in \Gamma(U_i, \mathcal{L}(\delta))$ as $f_i \cdot 1/f_i \in \mathcal{O}^*_{Y,y}$ $\forall y \in U_i$

$$\begin{array}{ccc} \mathcal{L}(\delta)|_{U_i} & \xrightarrow{\cdot f_i} & \mathcal{O}_{U_i} \\ \cup\!\!\mid & & \cup\!\!\mid \\ \underline{K(Y)} & \xrightarrow[f_i]{} & \underline{K(Y)} \end{array}$$
gives the inverse.

Thm. Let Y be an integral scheme.
$\delta \mapsto \mathcal{L}(\delta)$ gives a bijection

Thm. Let Y be an integral scheme.

1) The map $s \longmapsto \mathcal{L}(s)$ gives a bijection
$$\mathrm{Cart}(Y) \longleftrightarrow \{\text{invertible subsheaves of } \underline{K(Y)}\}.$$

2) The map in (1) induces an isom of groups
$$\mathrm{CaCl}(Y) \xrightarrow{\sim} \{\text{isom classes of invertible subsheaves of } \underline{K(Y)}\}.$$
where the group structure on the right again comes from $\otimes\, \mathcal{O}_Y$

3) The map induced by (2)

$$\begin{tikzcd}
\mathrm{CaCl}(Y) \arrow[r, "\sim"', "(2)"] \arrow[dr] & \{\text{isom classes of invertible subsheaves of } \underline{K(Y)}\} \arrow[d] \\
& \mathrm{Pic}(Y)
\end{tikzcd}$$

is an isom. of groups.

Pf. 1) Injection: Assume $s_\alpha, s_\beta \in \mathrm{Cart}(Y)$ have the same image. Choose an open covering
$$Y = \bigcup_{i \in I} U_i$$
such that $\exists\ f_i^\alpha, f_i^\beta$ for each i satisfying, $s_\alpha|_{U_i} = f_i^\alpha$, $s_\beta|_{U_i} = f_i^\beta$ $\forall i$.

Thus $\frac{1}{f_i^\alpha}\mathcal{O}_Y(U_i) = \frac{1}{f_i^\beta}\mathcal{O}_Y(U_i)$ $\forall i$

$\Rightarrow f_i^\beta / f_i^\alpha,\ f_i^\alpha / f_i^\beta \in \mathcal{O}_Y(U_i)$ $\forall i$

$\Rightarrow f_i^\beta / f_i^\alpha \in \mathcal{O}_Y(U_i)^*$ $\forall i$

$\Rightarrow s_\alpha = s_\beta \in \Gamma(Y, \underline{K(Y)^*}/\mathcal{O}_Y^*)$.

2) [illegible]: Given an invertible $\mathcal{O}_Y$ submod $\mathcal{F}$ of

Surjectivity: Given an invertible $\mathcal{O}_Y$ submod $\mathcal{F}$ of $\underline{K(Y)}$, choose a open covering $Y = \bigcup_{i \in I} U_i$ such that for each i $\mathcal{F}|_{U_i} = f_i \cdot \mathcal{O}_{U_i}$ for some $f_i \in K(Y)^*$

Since $f_i \cdot \mathcal{O}_{U_i}|_{U_i \cap U_j} = f_j \mathcal{O}_{U_j}|_{U_i \cap U_j}$

$f_i / f_j \in \mathcal{O}_{U_i}(U_i \cap U_j)^*$

$\Rightarrow \{ 1/f_i \in \Gamma(U_i, \underline{K(Y)^*}/\mathcal{O}_Y^*) \}_{i \in I}$ glue to give a section $s \in \Gamma(Y, \underline{K(Y)^*}/\mathcal{O}_Y^*)$

By claim $**$, $\mathcal{L}(s) = \mathcal{F}$.

2) By claim $**$, we have
$$\mathcal{L}(s_1) \otimes_{\mathcal{O}_Y} \mathcal{L}(s_2) \xrightarrow{\sim} \mathcal{L}(s_1 s_2)$$

So we only need to check: if $\mathcal{L}(s) \xrightarrow{\sim} \mathcal{O}_Y$, Then $\exists f \in K(Y)^*$ such that $s = f$ in $\Gamma(Y, \underline{K(Y)^*}/\mathcal{O}_Y^*)$.

To that end, fix an isom
$$\mathcal{O}_Y \xrightarrow{\sim} \mathcal{L}(s).$$

Denote the image of $1 \in \mathcal{O}_Y(Y)$ via this isom by $g \in K(Y)^*$.

So $\mathcal{L}(s)(U) = g \cdot \mathcal{O}_Y(U) \quad \forall U \underset{\text{open}}{\subseteq} Y$.

Choose an open cover $Y = \bigcup_{i \in I} U_i$ s.t $s|_{U_i} = f_i$ for some $f_i \in K(Y)^*$.

By claim $**$, $\mathcal{L}(s)(U_i) = 1/f_i \, \mathcal{O}_Y(U_i)$

Thus $1/f_i \, \mathcal{O}_Y(U_i) = g \cdot \mathcal{O}_Y(U_i) \quad \forall i$ $\qquad 1/f_i = g x$

$\Rightarrow g / 1/f_i \in \mathcal{O}_Y(U_i)^* \quad \forall i$ $\qquad g = 1/f_i \cdot y$

$\Rightarrow s = 1/g \in \Gamma(Y, \underline{K(Y)^*}/\mathcal{O}_Y^*)$ $\square$.

3) ... once we show every invertible ...

3) follows once we show every invertible $\mathcal{O}_Y$-mod is isom to some invertible sub $\mathcal{O}_Y$-mod of $\underline{K(Y)}$. Given an invertible $\mathcal{O}_Y$-mod $\mathcal{L}$, we produce an injective $\mathcal{O}_Y$-mod map $\mathcal{L} \to \underline{K(Y)}$. The image will be isom to the desired invertible $\mathcal{O}_Y$-submod $\mathcal{L}$.

To that end, choose an isom
$$\mathcal{L}_\eta \xrightarrow[\varphi]{\sim} K(Y)$$ where η is the gen pt of Y.

For $U \underset{\text{open}}{\subseteq} Y$, the canonical map composed with φ
$$\mathcal{L}(U) \longrightarrow \mathcal{L}_\eta \xrightarrow[\varphi]{\sim} K(Y)$$
gives a map of
$$\Psi_U : \mathcal{L}(U) \longrightarrow \mathcal{L}_\eta \xrightarrow{\sim} \underline{K(Y)}(U) = K(Y)$$
Ψ_U induces a $\mathcal{O}_Y$-lin map $\mathcal{L} \longrightarrow \underline{K(Y)}$.

Since Y is integral, $\mathcal{L}(U) \hookrightarrow \mathcal{L}_\eta$ and hence Ψ_U is inj $\forall U$. $\square$.